

NARUMALAR ACADEMY – ONLINE COACHING CENTRE

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COLLEGE TRB – MATHEMATICS

DEMO FILE

UNIT 1

1. Let G be a finite group acting transitively on a set X with $|X| >$

1. If the stabilizer G_x is abelian, then

A) G is abelian

B) G is solvable

C) G has a nontrivial normal abelian subgroup

D) G must be simple

Answer: C) G has a nontrivial normal abelian subgroup

Explanation: From the orbit–stabilizer theorem and properties of transitive primitive actions, G admits a normal subgroup isomorphic to a conjugate intersection of abelian stabilizers, hence nontrivial and abelian.

2. The number of automorphisms of the quaternion group Q_8 is

A) 24

B) 48

C) 8

D) 12

Answer: A) 24

Explanation: $\text{Aut}(Q_8)$ is isomorphic to S_4 , whose order is 24, using the action on the set $\{\pm i, \pm j, \pm k\}$ preserving quaternion multiplication structure.

3. Let p be prime. The number of elements of order p in

$GL(2, p)$ equals

- A) $p(p - 1)$
- B) $p(p + 1)$
- C) $(p^2 - 1)(p - 1)$
- D) $p^2 - 1$

Answer: B) $p(p + 1)$

Explanation: Elements of order p correspond to non-identity unipotent matrices $I + N$, where $N^2 = 0$ and $N \neq 0$; counting such N gives $p^2 - 1$ possibilities normalized by scalar equivalence.

4. If $f(x) = x^4 - 2$ over \mathbb{Q} , then the Galois group of f over \mathbb{Q} is

- A) C_4
- B) D_4
- C) V_4
- D) $C_2 \times C_4$

Answer: B) D_4

Explanation: Roots are $\sqrt{2}, i\sqrt{2}, -\sqrt{2}, -i\sqrt{2}$; the adjunction of i and $\sqrt{2}$ gives automorphisms forming a dihedral group D_4 .

5. For a finite abelian group G , the number of distinct irreducible representations over \mathbb{C} equals

- A) The exponent of G
- B) The rank of G
- C) $|G|$
- D) The number of cyclic direct factors

Answer: C) $|G|$

Explanation: Each group element defines a distinct 1-dimensional character since $\hat{G} \cong G$ for finite abelian groups.

6. Let $R = \mathbb{Z}[i]$. Then the number of units modulo the prime ideal $(1+i)$ is

- A) 2
- B) 4
- C) 1
- D) 3

Answer: C) 1

Explanation: Since $\mathbb{Z}[i]/(1+i) \cong \mathbb{Z}_2$, only 1 is a unit modulo $1+i$.

7. In \mathbb{F}_{p^n} , the number of elements whose minimal polynomial over \mathbb{F}_p has degree d equals

- A) $\phi(p^d - 1)$
- B) $d(p^d - p^{d-1})$
- C) $n\mu(d)$
- D) The number of monic irreducible polynomials of degree d multiplied by d

Answer: D) The number of monic irreducible polynomials of degree d multiplied by d

Explanation: Each irreducible polynomial contributes d roots, and the number of irreducibles of degree d over \mathbb{F}_p is given by

$$\frac{1}{d} \sum_{k|d} \mu(k) p^{d/k}.$$

8. For $f(x) = x^p - x - a \in \mathbb{F}_p[x]$ with $a \neq 0$, its splitting field over \mathbb{F}_p has degree

- A) 1
- B) p
- C) divides $p - 1$
- D) equal to the order of a in \mathbb{F}_p^\times

Answer: C) divides $p - 1$

Explanation: The extension generated by a root satisfies $\alpha^p - \alpha = a$, and changes under Frobenius map produce translations by elements of \mathbb{F}_p ; hence splitting field's degree divides $p - 1$.

9. In the ring $\mathbb{Z}_2[x]/(x^3 + x + 1)$, the multiplicative group of units has order

- A) 7
- B) 8
- C) 15
- D) 9

Answer: A) 7

Explanation: $x^3 + x + 1$ is irreducible over \mathbb{Z}_2 ; thus \mathbb{F}_{2^3} has multiplicative group of order $2^3 - 1 = 7$.

10. The determinant of the matrix representing the linear transformation $T(v) = Av$ on vector space V with matrix A satisfies

- A) $\det(T) = \text{trace}(A)$
- B) $\det(T) = |\lambda_1 + \lambda_2 + \cdots + \lambda_n|$
- C) $\det(T) = \prod \lambda_i$
- D) $\det(T) = \sum \lambda_i^2$

Answer: C) $\det(T) = \prod \lambda_i$

Explanation: Determinant equals the product of eigenvalues, invariant under similarity transformations.

11. The matrix of a normal operator on a complex inner product space is unitarily diagonalizable if and only if

- A) It is Hermitian
- B) It commutes with its adjoint
- C) It is idempotent
- D) It is orthogonal

Answer: B) It commutes with its adjoint

Explanation: Spectral theorem ensures that a normal operator A satisfying $AA^* = A^*A$ admits a unitary diagonalization.

12. The minimal polynomial of a linear operator coincides with its characteristic polynomial if and only if

- A) All eigenvalues are distinct
- B) The operator is diagonalizable
- C) The minimal polynomial is square-free
- D) Both A and B

Answer: D) Both A and B

Explanation: Distinct eigenvalues imply the minimal equals characteristic polynomial; equivalently, the operator is diagonalizable.

13. A quadratic form $q(x, y, z) = x^2 + y^2 + z^2$ over \mathbb{R} has matrix rank

- A) 2
- B) 1

- C) 3
- D) 0

Answer: C)3

Explanation: Three linearly independent squares correspond to full rank 3.

14. For a solvable group G of order 48, which of the following must be true?

- A) G has a normal Sylow 3-subgroup
- B) G is nilpotent
- C) G has a normal abelian subgroup of index 2
- D) G is simple

Answer: A) G has a normal Sylow 3-subgroup

Explanation: By Sylow theorems and solvability, at least one Sylow subgroup corresponding to prime with smallest index is normal.

15. Wedderburn's little theorem implies that every finite division ring is

- A) Commutative
- B) Simple non-commutative
- C) Isomorphic to a group algebra
- D) Quadratic in characteristic zero

Answer: A) Commutative

Explanation: Any finite division ring is a finite field, hence commutative.

16. Frobenius theorem on finite groups asserts that if G acts transitively with no fixed points except identity, then

- A) G is simple
- B) G has a normal complement called a Frobenius kernel
- C) G is nilpotent
- D) G is abelian

Answer: B) G has a normal complement called a Frobenius kernel

Explanation: The Frobenius kernel is a normal subgroup, complementing a subgroup (called the Frobenius complement).

17. The quaternion integers form a non-commutative ring that is

- A) Division ring
- B) Euclidean domain
- C) Non-commutative integral domain without zero divisors
- D) Neither integral domain nor division ring

Answer: C) Non-commutative integral domain without zero divisors

Explanation: Quaternions are non-commutative but possess multiplicative inverses for all nonzero elements.

18. According to Lagrange's theorem, the number of elements of order dividing n in a finite cyclic group of order n is

- A) $\varphi(n)$
- B) n
- C) Sum of all divisors of n
- D) $\mu(n)$

Answer: C) Sum of all divisors of n

Explanation: For each divisor $d \mid n$, $\varphi(d)$ elements have order d ; their sum equals n , the total elements.

19. The number of Sylow 7-subgroups in a group of order 168 must be

- A) 8
- B) 3

- C) 1
- D) 6

Answer: C) 1

Explanation: $168 = 2^3 \times 3 \times 7$; by congruence condition $n_7 \equiv 1 \pmod{7}$ and divides 24, thus only 1 possible, hence normal.

20. The polynomial $x^5 - x + 1$ over \mathbb{Q} has Galois group

- A) C_5
- B) S_5
- C) D_5
- D) A_5

Answer: B) S_5

Explanation: Irreducible quintic with non-square discriminant and rational root test fails; hence full symmetric group S_5 .

21. The dimension of an irreducible representation of a finite group divides

- A) The exponent of G
- B) The order of G
- C) $\phi(|G|)$
- D) The number of conjugacy classes

Answer: B) The order of G

Explanation: From basic representation theory of finite groups, degrees of irreducibles divide $|G|$.

22. Let T be a Hermitian operator on a complex inner product space. Then eigenvalues of T are

- A) Purely imaginary
- B) Real
- C) Complex conjugate pairs
- D) Unit modulus

Answer: B) Real

Explanation: Hermitian property ensures all eigenvalues are real due to $\langle Tx, x \rangle$ real for all x .

23. If A is unitary, then $\det(A)$ satisfies

- A) $|\det(A)| = 1$
- B) $|\det(A)| = 0$
- C) $\det(A)$ purely imaginary
- D) $\det(A) = \text{real number}$

Answer: A) $|\det(A)| = 1$

Explanation: A unitary matrix preserves inner product, implying product of eigenvalues has unit modulus.

24. The canonical form over \mathbb{C} of a nilpotent matrix of rank r with $n \times n$ dimension is

- A) Diagonal matrix with r nonzero entries
- B) Single Jordan block of size r
- C) Jordan form with zero diagonal and ones on superdiagonals
- D) Identity of order r

Answer: C) Jordan form with zero diagonal and ones on superdiagonals

Explanation: Nilpotent matrices' Jordan canonical form has zeros on diagonal and 1's just above diagonal representing chains.

25. For any quaternion $q = a + bi + cj + dk$, the norm $N(q)$ is

- A) $a^2 + b^2 + c^2 + d^2$
- B) $a^2 - b^2 - c^2 - d^2$
- C) $|a + b + c + d|$
- D) 1 if q is unitary

Answer: A) $a^2 + b^2 + c^2 + d^2$

Explanation: Product of quaternion with its conjugate yields $N(q) = a^2 + b^2 + c^2 + d^2$, central to the four-square theorem.